THE PRELIMINARY MEETING OF THE SOUTHWESTERN SECTION.

THE preliminary meeting of the proposed Southwestern Section* of the AMERICAN MATHEMATICAL SOCIETY took place at Columbia, Mo., on the 1st of December, 1906. About thirty-five persons, including the following members of the Society, were present:

Professor L. D. Ames, Mr. R. L. Börger, Professor A. S. Chessin, Brother Constantius, Professor G. R. Dean, Dr. E. L. Dodd, Dr. F. J. Dohmen, Mr. E. S. Haynes, Professor E. R. Hedrick, Professor O. D. Kellogg, Mr. W. A. Luby, Professor E. H. Moore, Professor Alexander Pell, Professor Oscar Schmiedel, Mr. F. C. Touton, Miss M. S. Walker, Dr. Paul Wernicke, Dr. W. D. A. Westfall, Mr. A. M. Wilson, Professor J. W. Withers.

The meeting was called to order by Professor Hedrick at 10:30 A. M. Professor Hedrick was elected chairman and Professor Chessin secretary of the meeting. Professor E. H. Moore was asked to preside as honorary chairman of the meeting for the afternoon session. The morning session adjourned at 1 P. M.

The meeting was called to order for the afternoon session at 2:30 P. M. The reading of papers was preceded by a short business meeting. On the motion made by Professor Chessin the members present unanimously selected St. Louis as the next meeting place of the proposed Southwestern Section.

A programme committee was elected, consisting of Professor E. R. Hedrick, chairman; Professor A. S. Chessin, secretary; Professor M. B. Porter.

At the conclusion of the meeting a motion was passed expressing the thanks of the members to all who had assisted in the formation of this section, especially to Professor E. H. Moore. The meeting adjourned at 5:45 P. M., and was followed by an informal supper at which about twenty were present.

* The formation of the Southwestern Section was authorized at the meeting of the Council on December 28.

The following papers were read:

(1) Professor L. D. AMES: "Notes on the orientation of a secant."

(2) Mr. R. L. BÖRGER: "On the determination of the subgroups of the special linear ternary homogeneous group in the Galois field p^2" (preliminary report).

(3) Professor A. S. CHESSIN: "On a particular case of motion of Sire's polytrope and Foucault's gyroscope."

(4) Professor G. R. DEAN: "The equations of motion of a compressible frictionless fluid" (preliminary report).

(5) Dr. E. L. DODD: "Incomplete double sequences as exemplified by infinitesimals."

(6) Professor S. EPSTEEN: "The probable error of a measurement a unit in length."

(7) Dr. O. E. GLENN: "On a class of operation groups of order $p_1^{m_1\mu_1}p_2^{m_2\mu_2}$."

(8) Professor O. D. KELLOGG: "Harmonic functions on the boundary of their region of definition."

(9) Professor OSCAR SCHMIEDEL: "The polynomial coefficient generalized and extended."

(10) Professor E. H. MOORE: "Homogeneous distributive functional operations of degree n."

(11) Professor H. B. NEWSON: "Real conformal transformations in space."

(12) Professor G. A. MILLER: "Groups in which every subgroup is either abelian or dihedral."

(13) Professor M. B. PORTER: "Change of variable in a multiple integral."

(14) Professor M. B. PORTER: "Pringsheim's criterion for expansibility in Taylor's series."

(15) Dr. PAUL WERNICKE: "On the interlocking and knotted curves and surfaces."

(16) Dr. W. D. A. WESTFALL: "Generalization of the fundamental theorem of Fourier constants."

(17) Professor E. H. MOORE: "Notes on Fourier's constants."

(18) Professor L. E. DICKSON: "Invariants of binary forms under modular transformations."

(19) Professor ALEXANDER PELL: "Some remarks on curves of constant torsion."

(20) Professor M. B. PORTER: "Polynomial convergents of a power series."

In the absence of the authors Dr. Glenn's paper was read by Professor Kellogg, Professor Dickson's by Professor Hedrick, Professor Porter's third paper by Dr. Westfall, and Professor Miller's paper and Professor Porter's first two papers were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In certain problems in the theory of potential it is desirable to be able to define uniquely the positive sense of a secant to a smooth curve, and also to define the angle it makes with a given line as a single valued continuous function of the points of intersection. Professor Ames showed that this is conveniently done by means of the definition of an angle from its sine and cosine taken together, as used in a previous paper (*American Journal of Mathematics*, October, 1905, page 349).

2. In Mr. Börger's paper the subgroups of order a power of p and those of order a multiple of p down to the order p^3M (M being prime to p) are determined for the special linear ternary homogeneous groups in a Galois field p^2.

3. When the mass of the rings in the apparatus is neglected the motion of Sire's polytrope or Foucault's gyroscope is expressible by means of elliptic or circular functions. Professor Chessin showed that the solution could be also effected in terms of elliptic functions in the case when $A_1 = A + C_1$; A_1, A_1, C_1 and A, A, C indicating respectively the principal moments of inertia of the internal ring and of the torus. Such a relation may be obtained by a proper construction of the gyroscope.

4. For the case of steady motion in two dimensions under no forces we have the equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

and the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{\rho}\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right).$$

Eliminating p and ρ by means of the isothermal relation $p = k\rho$, Professor Dean obtains an equation of the second order of Monge's form which is easily integrated, and the arbitrary functions may be determined to fit given boundary conditions. The adiabatic relation also gives a Monge equation.

If the acceleration due to gravity is introduced, another term is introduced into the equation, rendering the subsidiary system of pfaffians inexact, *i. e.*, not capable of being represented by a single integral equation.

In the case of motion in three dimensions, both the isothermal and adiabatic relations give equations in three independent variables analogous to Monge's. In the case of no forces the system of pfaffians is exact, but when forces are introduced the system does not satisfy the conditions of integrability.

5. Dr. Dodd notices that the totality of the infinitesimals a_{in}, when $n \doteq \infty$, forms an incomplete double sequence, which lacks the terms on one side of the main diagonal. A common error in the statement of a theorem regarding the substitution of infinitesimals in an infinite sum may be made clear by two sequences whose terms are: $a_{in} = 1/n$ and $\beta_{in} = 1/(n+1-i)$ respectively. In this case,

$$\lim_{n=\infty} \beta_{in}/a_{in} = 1\,; \lim_{n=\infty} \sum_{i=1}^{i=n} a_{in} = 1. \quad \text{But } \lim_{n=\infty} \sum_{i=1}^{i=n} \beta_{in} = \infty\,.$$

6. Basing his argument on an assumption which differs from the one used in Merriman's Least Squares, Professor Epsteen obtains a different value for the probable error of a measurement a unit in length, viz., $0.6745\sqrt{\sum pd^2(2n-1)^{-1}}$ where $p = 2(l + l')^{-1}$, and $d = l - l'$, l and l' being duplicate measurements.

7. The first part of Dr. Glenn's paper was published in full in the BULLETIN, May, 1906. In the second part he extends the methods there employed to include the determination of the defining relations of a class of groups of the given order, whose subgroup of order $p_1^{m_1\mu_1}$ is the abelian type $[\mu_1, \mu_1, \cdots, \mu_1]$, $(\mu_1 > 1)$. A general method of determining the number of distinct types in these relations is developed.

8. In this paper Professor Kellogg extends the investigations announced at the September meeting of the Society to

regions other than the circle. An apparently new view point is attained in seeking conditions for the existence of the derivatives on the boundary which bear as nearly as possible equally upon boundary curve and boundary values. Restricting the problem to normal derivatives the following result is obtained: Let $x = x(s)$, $y = y(s)$ be the parameter equations of the curve, s being the length of arc between a fixed and a varying point, and let $f(s)$ be the boundary values to be approached by the potential function. Then if three positive constants A, a and δ can be determined, such that for every s $|\Delta x'(s)| < A\,|\Delta s|^a$ and $|\Delta y'(s)| < A\,|\Delta s|^a$ as soon as $|\Delta s| < \delta$; and if the integral

$$\int \left| \frac{f'(s+t) - f'(s)}{t} \right| dt$$

is convergent for every value of s when extended over an interval containing $t = 0$ in its interior, then the derivative of the potential function formed in the direction of a normal to the boundary curve approaches a finite limit as the boundary is approached along the normal, and these limits form a continuous function on the boundary. Incidentally a generalization of Fredholm's proof of the Dirichlet principle is obtained, the boundary curve being subject only to the above restriction and to being a single closed curve without double points; and some new light is thrown upon the potentials of double distributions on a curve.

9. The binomial coefficient represented in Grassmann's notation by $n \cdot^a$ is generalized by Professor Schmiedel to embrace negative values of a and extended, the symbol $n \cdot^{a.b.c.\ldots.r}$ being introduced. The fundamental relations of the extended coefficients are derived and numerous applications are shown, notably the forming of the nth derivative of a function of several variables.

10. In generalization of current concepts of linear distributive functional operations Professor Moore proposes the following definition:

On the supposition that (a) the notations u relate to elements of a field F having no modulus; (b) the notations ϕ relate to elements of a class $\mathfrak{M}$ linear with respect to F, *i. e.*, such that the notations

$$u\phi = \phi u, \quad u_1\phi_1 + u_2\phi_2 = u_2\phi_2 + u_1\phi_1, \quad 1\phi = \phi, \quad 0\phi = 0,$$

$$(u_1+u_2)\phi = u_1\phi + u_2\phi, \; u(\phi_1+\phi_2) = u\phi_1 + u\phi_2, \; u_1(u_2\phi) = (u_1u_2)\phi$$

specify in each case definite elements of $\mathfrak{M}$; and (c) the notations ϕ^* relate to elements of a class $\mathfrak{M}^*$ linear with respect to F:

An operation A assigning to every ϕ a definite corresponding $\phi^* = A(\phi)$ is called a homogeneous distributive operation of degree n effective in a single valued way from $\mathfrak{M}$ to $\mathfrak{M}^*$ with respect to F, in case the $n+2$ elements

$$\psi_h^* = A(\psi_h) \qquad (h = 0, 1, \cdots, n+1)$$

of $\mathfrak{M}^*$ corresponding to $n+2$ elements

$$\psi_h = u_{h1}\phi_1 + u_{h2}\phi_2 \quad (h = 0, 1, \cdots, n+2)$$

in a binary linear system (ϕ_1, ϕ_2) in $\mathfrak{M}$ satisfy always the following relation:

$$(A_2) \qquad \begin{vmatrix} \psi_0^* & u_{01}^n & u_{01}^{n-1}u_{02} & \cdots & u_{02}^n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \psi_{n+1}^* & u_{n+11}^n & u_{n+11}^{n-1}u_{n+12} & \cdots & u_{n+12}^n \end{vmatrix} = 0.$$

It follows that there exist in an unique way single valued operations

$$A_{\phi_1, \phi_2}^{n_1, n_2} \quad (n_1, n_2 = 0, 1, \cdots, n; \; n_1 + n_2 = n)$$

on two arguments ϕ_1, ϕ_2 from $\mathfrak{M}$ to $\mathfrak{M}^*$ such that identically

$$(B_2) \qquad A(u_1\phi_1 + u_2\phi_2) = \sum_{\substack{n_1, n_2 = 0 \\ n_1 + n_2 = n}}^{n} \frac{n!}{n_1!\, n_2!} u_1^{n_1} u_2^{n_2} A_{\phi_1, \phi_2}^{n_1, n_2}.$$

Conversely, the existence of such operations with the identity (B_2) implies that relation (A_2) holds. Further similar identities (A_s), (B_s) hold for every s-ary linear system $(\phi_1, \cdots, \phi_s)$.

The operations entering these identities are simply expressible in terms of the operation

$$A_{\phi_1, \cdots, \phi_n}^{1, \cdots, 1}$$

on n arguments occurring in (B_n); this operation is symmetric

on its n arguments and linear distributive on each argument, and

$$A(\phi) = A^{1,\cdots,1}_{\phi_1,\cdots,\phi_n}.$$

Conversely, if $A(\phi_1, \cdots, \phi_n)$ is an operation symmetric on its n arguments and linear distributive on each, then

$$A(\phi) = A(\phi, \cdots, \phi)$$

is an operation on ϕ homogeneous distributive of degree n.

11. Recently Professor Ugo Amaldi has published in the *Memorie* of the Turin Academy a paper on the continuous groups of real conformal transformations in ordinary space. The groups were investigated by the methods of Lie and expressed in the usual infinitesimal notation. In Professor Newson's paper these real conformal transformations are studied by pure geometric methods and all types of such transformations are found. The subgroups of the real conformal group are determined and classified according to their types. Most of Professor Newson's results have been in his possession about eight years, but the investigation was completed only recently. The paper has been accepted for publication by the *Giornale di Matematiche*.

12. The first part of Professor Miller's paper is devoted to a proof of the theorem that a group is necessarily solvable if every non-abelian subgroup is dihedral. This theorem is then extended so as to read: if every non-abelian subgroup of a group is either dihedral or associate dihedral, the group is solvable. It is observed that every non-abelian subgroup of a dihedral group is dihedral, and all the non-dihedral groups in which every non-abelian subgroup is dihedral are investigated. It is proved that if such a group is non-abelian it must belong to one of the following five types: (1) the direct product of the octic group and a group of order p, p being any prime number; (2) the direct product of the dihedral group of order $2q$, q being an odd prime, and a group of order p; (3) the direct product of the dihedral group of order $4q$ and a group of order 2; (4) the group obtained by extending the abelian group of order p^2, $p > 2$, and of type (1, 1) by means of an operator of order 2 which transforms each operator of this abelian group into its inverse; (5) the group of order $4q$, q

being prime and $\equiv 1 \bmod 4$, which is contained in the holomorph of the cyclic group of order q. The first three types contain invariant operators while there is no invariant operator besides identity in either of the last two types.

13. Professor Porter's first paper was a pedagogic note in which attention was called to the fact that if, for example, in three dimensions, we take as our element of volume

$$\Delta = \frac{1}{3!}\begin{vmatrix} \Delta_1 x & \Delta_2 x & \Delta_3 x \\ \Delta_1 y & \Delta_2 y & \Delta_3 y \\ \Delta_1 z & \Delta_2 z & \Delta_3 z \end{vmatrix}$$

and make the one to one change of variable $x = x(u, v, w)$, etc., we get

$$(1) \qquad \Delta = \frac{1}{3!} J \cdot \begin{vmatrix} \Delta_1 u & \Delta_2 u & \Delta_3 u \\ \Delta_1 v & \Delta_2 v & \Delta_3 v \\ \Delta_1 w & \Delta_2 w & \Delta_3 w \end{vmatrix} (1 + t)$$

$$= \frac{1}{3!} J \Delta'(1 + t),$$

where $\lim_{\Delta' = 0} t = 0$, provided the jacobian J of the transformation is continuous.

If the jacobian does not change sign, the tetrahedron Δ' will not overlap if the tetrahedra Δ do not by (1). The usual formulas for change of variable in triple integrals follow at once from Duhamel's principle.

14. Professor Porter's second paper contains further results of his research which began in the current number of the *Annals of Mathematics* and will be continued in the same journal.

15. The purpose of Dr. Wernicke's paper is to extend the applicability of Gauss's criterion for the interlocking of curves. In the double integral

$$\frac{1}{4\pi} \int\int S \frac{\rho - \rho'}{r^3} d\rho d\rho'$$

S indicates the scalar product of the vectors, the curves C, C' being given by their vector equations $\rho = \rho(t)$, $\rho' = \rho'(t)$; and

r is the length of $\rho - \rho'$. This integral is an integer m whenever C coils m times about C' in a fixed positive direction. Any bilateral simply connected surface bounded by C' is pierced at least $|m|$ times by C. It was shown that when $m = 0$, C may still so coil about C' and about itself that it cannot be "disentangled." A surface bounded by C may be so constructed that C' does not pierce it. On such a surface, however, an auxiliary curve may be drawn so as to interlock with C' in the former way. Both ways of interlocking may occur simultaneously in the case both of the original and auxiliary curves, if the latter be properly chosen; but a repeated application of Gauss's criterion will reveal all these relations. Moreover, for n-space, the criterion may be so modified as to characterize the interlocking of closed k-surfaces with $(n-k-1)$-surfaces, where $k = 0, 1, \cdots, n-1$. Also, while the double integral is not immediately applicable to a curve interlocking with itself, or forming a "knot," it may be used to reveal knottedness, by extending it over the given curve and another accompanying it at a sufficiently small vectorial distance.

16. Dr. Westfall showed, by means of a recent theorem of Stekloff, that the fundamental theorem of Fourier's constants,

$$\int_a^b A(x) f^2(x) dx = \sum_1^\infty a_i^2, \qquad a_i = \int_a^b A(x) f(x) \phi_i(x) dx,$$

holds for any limited and integrable function $f(x)$ and Hilbert's characteristic functions $\phi_i(x)$ of a self adjoined differential equation. It follows that, if $A(x)$ does not vanish in (a, b), two limited and integrable functions having the same generalized Fourier's constants differ by an integrably null function in (a, b), and if $\sum_1^\infty a_i \phi_i(x)$ converges to an integrable and finite function in any sub-interval (a, β), or (a, b), it likewise differs from $f(x)$ in that sub-interval by an integrally null function.

17. Professor Moore's note on the Fourier constants of an integrable function appears elsewhere in this number of the BULLETIN.

18. Although linear transformations with a prime modulus p are employed extensively in many branches of mathematics, the invariants of quantics under such transformations seem to have escaped attention. The first steps in the construction of a the-

ory of such invariants are made in the paper by Professor Dickson. The subject is at once seen to be much more difficult than the ordinary algebraic theory. For instance, the terms of an invariant need not be of the same degree nor of constant weight. Again, the annihilators are far more complicated, involving partial derivatives of higher than the first order. Direct computation is necessarily a luxury in this field.

As the fundamental invariants of $ax^2 + 2bxy + cy^2$, when $p > 2$, we may take the discriminant Δ and

$$J = (a^\mu + 1)(c^\mu + 1)\left\{ \sum_{i=0}^{\mu} a^i c^i b^{2\mu - 2i} - 1 \right\},$$

where $\mu = \frac{1}{2}(p-1)$. Then $\Delta J = 0$ and $(J+1)^2 = I + 1$, where

$$I = (a^{p-1} - 1)(b^{p-1} - 1)(c^{p-1} - 1).$$

Replacing p by p^n, we obtain results valid for the Galois field of order p^n. Every quantic has an invariant analogous to I, viz., $\pi(a_i^{p^n-1} - 1)$, the product extending over all the coefficients a_i of the quantic.

A cubic form, for prime modulus $p > 3$, has an absolute invariant K of constant degree $p - 1$ containing $2m$ terms, where m is the number of partitions of $p - 1$ into $3l_0 + 2l_1 + l_2$. Let Δ denote -3 times the algebraic discriminant $a_0^2 a_2^2 + \cdots$ of the cubic form (with binomial coefficients). Then $(\Delta^{\frac{1}{3}(p+1)} - \Delta)K = 0$.

As in these illustrations on quadratic and cubic forms, so in general the new invariants have simple relations with the ordinary algebraic invariants. These results will be incorporated in a paper offered to the *Transactions*.

19. Professor Pell showed that any curve of constant torsion may be considered as the locus of the middle points of the chords connecting corresponding points on a minimal curve and a spherical curve, correlated to this minimal curve. This theorem was obtained from Lie's formulas for constructing a minimal surface passing through a given contour, when the directions of the normals along this contour of the minimal surface are given.

20. For any power series $\sum_0^\infty a_n x^n$, with a radius of convergence r, such that both $\lim_{n=\infty} a_n x^n$ becomes definitely infinite and $|a_n^{-1} x^{-n} \sum_0^n a_n x^n| < L$ while $|x| > r + e$, Professor

Porter showed that there were points on the circle of convergence which are condensation points for the zeros of the polynomial convergents and that in the neighborhood of every point of this circle the set of these polynomials took on values less than any assigned number however small. He also showed that no set of these polynomials remained limited throughout any region lying outside the circle of convergence.

A. S. CHESSIN,
Secretary.

SELECTED TOPICS IN THE THEORY OF BOUNDARY VALUE PROBLEMS OF DIFFERENTIAL EQUATIONS.

AN ABSTRACT OF FOUR LECTURES DELIVERED AT THE NEW HAVEN COLLOQUIUM, SEPTEMBER 5–8, 1906.

BY PROFESSOR MAX MASON.

§ 1.

FUNCTIONAL equations — particularly integral equations — have aroused much interest and activity in recent years. Important contributions to the subject have been made, notably by Volterra, Fredholm and Hilbert, and the results have found application in the field of mathematical physics and differential equations. The equations studied have been for the most part of a type difficult of solution, and the treatment correspondingly complicated.

There is however a type of functional equation whose solution may be obtained in a simple manner. Consider the equation

$$f = g + Sf, \tag{1}$$

where g is a known function, and S a *linear operator*, that is $S(u + v) = Su + Sv$. *The operator S will be called convergent if the infinite series*

$$\phi + S\phi + S^2\phi + S^3\phi + \cdots$$

converges for all functions ϕ which satisfy the conditions of con-

tinuity demanded of g and f, and if the convergence is such that the operation S may be performed on this series term by term.

Suppose S is convergent, and that a solution f of (1) exists. Then

$$\begin{aligned} f &= g + Sf = g + S(g + Sf) \\ &= g + Sg + S^2 f = g + Sg + S^2(g + Sf) \\ &= g + Sg + S\,g + S^3 f = \cdots, \end{aligned}$$

the equation

$$f = g + Sg + S^2 g + S^3 g + \cdots + S^{n-1} g + S^n f$$

being obtained after $n - 1$ substitutions of $g + Sf$ for f. If S is convergent, the limit for $n = \infty$ may be taken. It follows that if a solution of (1) exists it has the form

$$f = g + Sg + S^2 g + S^3 g + \cdots. \tag{2}$$

Conversely, if S is a convergent operator the function defined by (2) is a solution of (1), for

$$\begin{aligned} Sf &= Sg + S^2 g + S^3 g + \cdots \\ &= f - g. \end{aligned}$$

To summarize: *If S is a convergent operator, the equation* (1) *admits a unique solution, given by equation* (2).

A simple example is furnished by the equation

$$f(x) = g(x) + \int_a^x K(x, y) f(y)\,dy. \tag{3}$$

The operator S is convergent. For if Φ be the maximum of the absolute value of a function ϕ, and κ the maximum of K, then

$$| S^n \phi | \leqq \frac{(x - a)^n \kappa^n}{n!} \Phi.$$

These results will be applied to the study of certain boundary value problems of differential equations. This method has some advantages over the method of successive approximations, which has been used with such success, notably by Picard. The at-

tention is at once directed to the essential difficulty, that of convergence, and the uniqueness of the solution is proved without special investigation. The method is applicable however only to linear equations.

§ 2.

The linear partial differential equation of the second order in two independent variables

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

is said to be of elliptic, hyperbolic, or parabolic type, according as $AC - B^2$ is positive, negative, or zero, in the region considered. The normal forms of these types are

$$u_{xx} + u_{yy} = au_x + bu_y + cu + f,$$

$$u_{xy} = au_x + bu_y + cu + f, \quad u_{xx} = au_x + bu_y + cu + f,$$

the general equation being reduced to the normal form of the type to which it belongs, by a transformation of the independent variables.

The most important representative of the elliptic type is the potential equation

$$u_{xx} + u_{yy} = 0, \tag{4}$$

and the boundary value problems for this equation are fundamental in the treatment of similar problems for the general equation of elliptic type. Dirichlet's problem — to determine a solution of (4) within a closed region Ω which assumes given values on the boundary of Ω — is the most famous of boundary value problems. *There exists a unique solution of this problem.* The proofs of this statement given by Schwarz, Neumann, and Poincaré are classic. In case the region Ω is convex, Neumann's method gives the simplest solution. The results of § 1 may be applied to interpret this method. The function

$$u(x, y) = \frac{1}{\pi}\int_0^t f(t)\frac{\partial}{\partial t}\Theta[x, y, \xi(t), \eta(t)]dt, \quad \Theta = \arctan\frac{\eta(t) - y}{\xi(t) - x},$$

containing the arbitrary function $f(t)$, is a solution of (4) at all points within Ω, the boundary of Ω being given by the curve

$x = \xi(t), y = \eta(t)$. This function $f(t)$ must be so determined that u takes the value $g(s)$ at the point $x = \xi(s)$, $y = \eta(s)$. On account of the discontinuity of u in crossing the boundary this leads to the following integral equations for f:

$$f(s) = g(s) - \frac{1}{\pi}\int_0^l f(t)\frac{\partial\Theta(s,t)}{\partial t}dt, \tag{5}$$

where $\Theta(s, t) = \Theta(\xi(s), \eta(s), \xi(t), \eta(t))$. This equation is of the type considered in § 1, but the operator is not convergent. However, on adding $f(s)$ to both sides and dividing by two, the equation takes the form

$$f(s) = \tfrac{1}{2}g(s) + Sf,$$

where

$$Sf = -\frac{1}{2\pi}\int_0^l \{f(t) - f(s)\}\frac{\partial\Theta(s,t)}{\partial t}dt.$$

The convergence proof of Neumann's method shows that the operator S is convergent. Hence a unique solution of (5) exists and is given by

$$f = \tfrac{1}{2}\{g + Sg + S^2g + \cdots\}.$$

The function u produced by this function f is the desired solution of Dirichlet's problem.

On account of the above existence theorem, the existence of a Green's function is assured. This function has the form

$$G(x, y, \xi, \eta) = \log\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} + g(x, y, \xi, \eta),$$

where (ξ, η) is a parameter point within Ω, and G vanishes identically in ξ, η when the point (x, y) lies on the boundary of Ω. By the aid of this function a solution of the equation

$$u_{xx} + u_{yy} = f(x, y) \tag{6}$$

which assumes the values $g(s)$ on the boundary of Ω is given by the formula

$$u = u_0 - \frac{1}{2\pi}\int\int_\Omega G(x, y, \xi, \eta)f(\xi, \eta)d\xi d\eta, \tag{7}$$

where u_0 is the solution of Dirichlet's problem for the boundary values $g(s)$, provided that f satisfies a certain continuity condition.

A more general equation of elliptic type, and one of frequent occurrence in the applications, is

$$v_{xx} + v_{yy} = cv + f. \tag{8}$$

Replacing f in equation (6) by $cv + f$ it is seen that the solution of (8) which assumes the boundary values $g(s)$ is given by the solution of the integral equation

$$v = u - \frac{1}{2\pi}\int\int_\Omega G(x, y, \xi, \eta)c(\xi, \eta)v(\xi, \eta)d\xi d\eta, \tag{9}$$

where u is a known function, given by equation (7). It may be easily shown that the operator S in this equation is convergent, if the region Ω be sufficiently small, the proof depending on the fact that

$$\int\int_\Omega Gd\xi d\eta$$

approaches zero with the area of Ω. *There exists one and only one solution of the boundary value problem of equation* (8) *if the region* Ω *is sufficiently small.*

The general equation of elliptic type,

$$u_{xx} + v_{yy} = av_x + bv_y + cv + f \tag{10}$$

presents greater difficulties. To determine a solution of the boundary value problem, a functional equation of the form

$$v = u - \frac{1}{2\pi}\int\int_\Omega G(av_x + bv_y + cv)d\xi d\eta \tag{11}$$

is to be solved. The operator S involves differentiation as well as integration. The proof that S is convergent is accordingly more difficult, and an investigation regarding continuity is necessary to show that a solution of (11) is also a solution of (10); the results are however the same as in the case of equation (8). This existence theorem was first obtained by Picard, who used the method of successive approximations. The restriction on the size of the region in the theorem is not

one due to method alone. In point of fact, if the region be allowed to increase in size, a position is reached for which the theorem is no longer true. Some of the most interesting investigations of special boundary value problems deal with the equation

$$v_{xx} + v_{yy} = \lambda c v,$$

where λ is a parameter, the region Ω being fixed. The results are analogous to those obtained in § 4 for a similar problem in ordinary differential equations.

To Picard is due a remarkable theorem regarding the nature of solutions of the differential equation (10), viz. : *Every equation of elliptic type whose coefficients are analytic functions possesses only analytic solutions*; even though the solution assumes non-analytic boundary values, it will be analytic inside the region.

§ 3.

The equations of hyperbolic type are especially susceptible to treatment by the method of § 1. By the aid of this method a boundary problem may be solved which includes as special cases many problems whose solutions were originally obtained by quite distinct methods.

The equation

$$u_{xy} = au_x + bu_y + cu + f \tag{12}$$

is to be solved under the conditions

$$(u)_{y=\phi(x)} = U(x), \qquad (u_y)_{x=\psi(y)} = Y(y),$$

where ϕ, ψ, U, Y, are given functions.

Let $u_x = v$. The function

$$u = \int_{\phi(x)}^{y}\int_{\psi(\eta)}^{x} v(\xi, \eta)d\xi d\eta + \int_{\phi(x)}^{y} Y(\eta)dy + U(x)$$

satisfies the boundary conditions, whatever v be. On substituting this value u in (12), the equation becomes a functional equation for the determination of v, of the form

$$v = g + Sv,$$

where S is of somewhat complicated form, but involves in-

tegration only, the upper limits of each integral being x or y. On account of this fact the proof that S is a convergent operator may be made without difficulty, however large the region within which the point (x, y) may vary. The proof is analogous to that given in connection with equation (3). *It follows that there exists one and only one solution of the boundary problem.* If ϕ and ψ are inverse functions, so that $x = \psi(y)$ and $y = \phi(x)$ define the same curve, the problem is equivalent to finding a solution u of (12) when u and the normal derivative of u are given on a monotonic curve. If $\phi(x) = b$, $\psi(y) = a$, the boundary condition is equivalent to giving the values of u on the lines $x = a$, $y = b$. Other special cases may be easily obtained.

It may be readily seen from the expression for u in terms of v that the continuity properties of the functions ϕ, ψ, U, Y determine to a large extent the continuity properties of u throughout its region of definition. The equations of hyperbolic type thus differ fundamentally from the equations of elliptic type. In fact, S. Bernstein has recently proved that every equation of hyperbolic type possesses non-analytic solutions.

The general equation of parabolic type offers exceptional difficulties. But little is known of the nature of the solutions or of the boundary conditions which may be placed upon them. These equations form a limiting case between those of elliptic and hyperbolic types. Practically nothing is known regarding the boundary problems of equations which are of one type in a part of the region considered and of another in the remainder.

§ 4.

Many of the boundary value problems for partial differential equations may be reduced by the familiar substitution $u = X(x) \cdot Y(y)$, to boundary value problems for ordinary differential equations. The differential equation

$$y'' + \lambda A(x) y = 0, \tag{13}$$

where λ is a parameter, is typical of a large class, and has formed the subject of numerous investigations. The fundamental existence theorem for this equation may be easily proved by the method of § 1. The function

$$y = \int_a^x (x - \xi) u(\xi) d\xi + \alpha'(x - a) + \alpha$$

satisfies the conditions $y(a) = \alpha$, $y'(a) = \alpha'$, whatever be u. It is a solution of (13) if u is a solution of the integral equation

$$u = -\lambda A(x)[\alpha'(x-a) + \alpha] - \lambda A(x)\int_a^x (x-\xi)u(\xi)d\xi,$$

which is a special case of (3). There exists accordingly one and only one solution of (13) satisfying the initial conditions. From the form of the solution it is seen immediately that y is an integral transcendental function of the parameter λ.

A typical boundary problem for (13) is to determine a solution, not identically zero, which vanishes for two given values a and b of x. The general solution of (13) having the form

$$y = c_1\eta_1 + c_2\eta_2,$$

this problem may be solved when and only when λ is a zero of the transcendental function

$$\delta(\lambda) = \begin{vmatrix} \eta_1(a) & \eta_2(a) \\ \eta_1(b) & \eta_2(b) \end{vmatrix}.$$

This problem was first studied by Sturm, and his results may be summarized in the following theorem — Sturm's theorem of oscillation: *If $A(x)$ does not change sign the parameter λ may be determined in one and only one way so that a solution of (13) exists which vanishes at a and at b, and n times between a and b.*

This result has since been proved by various methods, some of which are capable of greater generalization than the original methods of Sturm. The problem may be reduced to the solution of an integral equation. Any solution of the equation

$$y(x) = \lambda\int_a^b A(\xi)G(x, \xi)y(\xi)\,d\xi,$$

where

$$G(x, \xi) = \tfrac{1}{2}\left\{|x-\xi| - \frac{(b-x)(\xi-a) + (b-\xi)(x-a)}{b-a}\right\},$$

is a solution of (13) and vanishes at a and b. This integral equation is of a type studied by Fredholm and Hilbert. If

$A(x)$ does not change sign, it can be reduced to the form

$$\eta(x) = \lambda \int_a^b K(x, \xi)\eta(\xi)d\xi,$$

where K is symmetric, the only case to which Hilbert's results apply.* The application of these results proves immediately the existence of an infinite series of "normal parameter values" — zeros of $\delta(\lambda)$ — for each of which there exists a "normal function," a solution of the boundary value problem. Hilbert proves further that any function may be expanded in terms of these normal functions, if it is continuous together with its first and second derivatives. The most general result regarding the expansion of an arbitrary function have however been attained by Kneser, using a different method.

If the function $A(x)$ changes sign, the above method does not apply. The existence theorem may however be obtained by a method based on the consideration of certain minimum problems. Interesting properties of the normal functions are thus set in evidence, and by the aid of these properties a very simple proof for the expansion of a function in terms of normal functions is obtained, even in the case that $A(x)$ changes sign.

The field of boundary value problems is of enormous extent, and comparatively little is known territory. While the theorems stated above are typical, interesting variations result by generalization to more variables, higher derivatives, and more general boundary conditions. Some of these generalizations have been rigorously made, many merely guessed. In particular, the boundary value problems for systems of differential equations offer a wide field for research, and work in this field would be of especial value in the applications.

SHEFFIELD SCIENTIFIC SCHOOL,
YALE UNIVERSITY.

* Since the above was written a fifth installment of Hilbert's Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen has appeared, in which the case where the function K is not symmetric is considered. (*Göttinger Nachrichten*, 1906, p. 439.)

NOTE ON FOURIER'S CONSTANTS.

BY PROFESSOR E. H. MOORE

(Read at the Preliminary Meeting of the Southwestern Section of the American Mathematical Society, December 1, 1906.)

A FUNCTION $f(x)$ real and single valued, bounded, and in the sense of Riemann integrable from 0 to 2π gives rise to the Fourier's series

$$(1)\quad \tfrac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) = \tfrac{1}{2}\sum_{k=-\infty}^{+\infty}(a_k \cos kx + b_k \sin kx),$$

where

$$(2)\qquad a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\, dx, \qquad (k = 0, \pm 1, \cdots)$$
$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\, dx.$$

Hurwitz * calls the constants a_k, b_k the Fourier's constants of the integrable function $f(x)$, and for the realm of integrable functions he studies the theory of Fourier's constants instead of the theory of Fourier's series which are convergent only under conditions.

The realm of integrable functions is a realm of integrity, *i. e.*, the sum, the difference, and the product of two integrable functions is integrable. The constants a_k, b_k, or in more explicit notation a_{kf}, b_{kf}, depend in a linear distributive way on the function $f = f(x)$. Further, the constants a_{kh}, b_{kh}, for the product $h(x) = f(x)\,g(x)$ of two functions $f(x)$, $g(x)$ are determinable from the constants a_{kf}, b_{kf} and a_{kg}, b_{kg} of these functions. In fact, from (2) we have

$$(3)\qquad a_{0f} = \frac{1}{\pi}\int_0^{2\pi} f(x)dx,$$

$$(4)\qquad a_{kf} = a_{0,f(x)\cos kx}, \quad b_{kf} = a_{0,f(x)\sin kx},$$

* Hurwitz, "Ueber die Fourierschen Konstanten integrierbarer Funktionen," *Math. Annalen*, vol. 57 (1903) pp. 425–446, and vol. 59 (1904), p. 553.

so that

$$(5) \qquad a_{kh} = a_{0,\,fg\cos kx}, \quad b_{kh} = a_{0,\,fg\sin kx}.$$

These formulas in effect make the expression of a_{kh}, b_{kh} depend simply upon the expression of a_{0h} in terms of the constants of f and g. The fundamental theorem of the theory of Fourier's constants states that

$$(6) \qquad a_{0h} = \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx = \tfrac{1}{2}a_{0f}a_{0g} + \sum_{k=1}^{\infty}(a_{kf}a_{kg} + b_{kf}b_{kg}).$$

If either $f(x)$ or $g(x)$ is expressible as a uniformly convergent Fourier series, the relation (6) follows by termwise integration. In its generality the fundamental theorem with important applications is due to de la Vallée Poussin * (1893), and it has been extended (1903) by Stekloff to all the types of Fourier constants frequently employed in analysis in connection with the representation of arbitrary functions of one or of several variables.

Hurwitz (loc. cit., volume 57, page 437) expresses the constants of $h(x) = f(x)\,g(x)$ in terms of those of $f(x)$, $g(x)$, apart from notation, as follows:

$$a_{0h} = \tfrac{1}{2}a_{0f}a_{0g} + \sum_{k=1}^{\infty}(a_{kf}a_{kg} + b_{kf}b_{kg}),$$

$$(7) \qquad a_{nh} = \tfrac{1}{2}a_{0f}a_{ng} + \tfrac{1}{2}\sum_{k=1}^{\infty}\{a_{kf}(a_{k+n\,g}+a_{k-n\,g}) + b_{kf}(b_{k+n\,g}+b_{k-n\,g})\},$$

$$b_{nh} = \tfrac{1}{2}a_{0f}b_{ng} + \tfrac{1}{2}\sum_{k=1}^{\infty}\{a_{kf}(b_{k+n\,g}-b_{k-n\,g}) - b_{kf}(a_{k+n\,g}-a_{k-n\,g})\}.$$

Hurwitz remarks further that these series converge absolutely, even when the terms at present collected into binomial expressions are taken as the constituent terms of the series. In the formulas (7) the rôles of f and g may of course be interchanged.

The series on the right of the equations (7) should however be expressed symmetrically with respect to f and g.

Let us use as Fourier constants

$$(2) \qquad A_{kf} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos kx\,dx, \qquad B_{kf} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin kx\,dx,$$

* For references see the memoirs of Hurwitz

viz., the halves of the constants according to Hurwitz. Then with use of the relations

$$(\bar{2}^\circ) \qquad A_{-k,l} = A_{k,l}, \qquad B_{-k,l} = -B_{k,l}$$

one readily finds from (7)

$$(7) \qquad \begin{aligned} A_{nh} &= \sum_{k+l=n}^{k,\,l=0,\,\pm 1,\,\cdots} (A_{k,l}A_{l,g} - B_{k,l}B_{l,g}), \\ B_{nh} &= \sum_{k+l=n}^{k,\,l=0,\,\pm 1,\,\cdots} (A_{k,l}B_{l,g} + B_{k,l}A_{l,g}). \end{aligned} \qquad (n = 0, \pm 1, \cdots)$$

These equations ($\bar{7}$) have the desired symmetric form. The form suggests moreover the introduction of the complex Fourier constants

$$(\bar{2}_a) \qquad C_{k,l} = A_{k,l} + jB_{k,l} \qquad (j^2 = -1),$$

in terms of which we have

$$(\bar{7}_a) \qquad C_{nh} = \sum_{k+l=n}^{k,\,l=0,\,\pm 1,\,\cdots} C_{k,l}C_{l,g} \quad (n = 0, \pm 1, \cdots).$$

This result may be expressed thus:

To every integrable function $f(x)$ there corresponds a system of Fourier constants $A_{k,l}$, $B_{k,l}$, $C_{k,l}$ $(2, \bar{2}_a)$ and a Laurent series with coefficients $C_{k,l}$

$$f(x) \sim \sum_{k=-\infty}^{+\infty} C_{k,l}Z^k.$$

This Laurent series is to be considered formally. Then the Laurent series corresponding to the product of two functions is the formal product of the two Laurent series corresponding respectively to the two functions.

As thus expressed, the fundamental theorem of the theory of Fourier's constants points to the known expression of Fourier series for suitably conditioned functions as convergent Laurent series.

THE UNIVERSITY OF CHICAGO,
December 7, 1906.

ON THE MINIMUM NUMBER OF OPERATORS WHOSE ORDERS EXCEED TWO IN ANY FINITE GROUP.

BY PROFESSOR G. A. MILLER.

LET the order of any group G be represented by $g = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}$; $p_1, p_2, \cdots, p_\lambda$ being distinct odd prime numbers. When $\alpha_0 = 0$ all the operators of G except identity are of orders which exceed two. Hence we shall assume in what follows that $\alpha_0 > 0$. In this case it is easy to see that there is at least one group of order g in which the number of operators whose orders exceed two is exactly

$$\frac{(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda} - 1)g}{2p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}} \equiv N.$$

Such a group may be constructed by forming the direct product of a group of order $2p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}$ in which just half the operators are of order two * and the abelian group of order $2^{\alpha_0 - 1}$ and of type $(1, 1, 1, \cdots)$. The main objects of this paper are to prove that N is the minimum number of operators whose orders exceed two that can occur in a group of order g, and if a group of this order has exactly N operators whose orders exceed two it is the direct product of a group in which just half the operators are of order two and the abelian group of order $2^{\alpha_0 - 1}$ and of type $(1, 1, 1, \cdots)$.

When $g = 2^{\alpha_0}$, $N = 0$ and the theorem requires no proof. When $\alpha_0 = 1$, $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda} - 1$ and the theorem is evident from the well-known theorem that every group whose order is twice an odd number contains a subgroup of half its order which is composed of operators of odd order in addition to identity. Hence we shall assume in what follows that $\alpha_0 > 1$ and $N > 0$. Since G is supposed to contain a minimum number of operators whose orders exceed two, it may be assumed that over half of the operators of G are of order two. Hence G is generated by its operators of order two, and as it contains oper-

* If just half the operators of a group are of order 2, the order of the group is twice an odd number, and all its operators of odd order together with identity constitute an abelian subgroup whose order is half the order of the group.

ators whose orders exceed two, it must be non-abelian. It must, therefore, contain non-invariant operators of order two.

Let H_1 be the subgroup of G which is composed of all the operators of G which are commutative with one of its non-invariant operators t_1 of order two. The products obtained by multiplying t_1 into operators of order two in $G - H_1$ are of orders which exceed two, since all of these operators of order two are non-commutative with t_1. Hence $G - H_1$ contains at least as many operators whose orders exceed two as of order two. From this it follows that more than half the operators of H_1 are of order two. If H_1 contains operators whose orders exceed two it is non-abelian and vice versa.

When H_1 is non-abelian, it contains a non-invariant operator t_2 of order two. In this case its subgroup composed of its operators which are commutative with t_2 may be denoted by H_2. More than half the operators of H_2 are of order two; and, if it is non-abelian, we continue in the same way until we arrive at an abelian subgroup H_m. All the operators of H_m except identity are of order two. Since at least half of the operators of $H_x - H_{x+1}$ $(x = 1, 2, \cdots, m-1)$ are of orders which exceed two, it follows that at least half the operators of $G - H_m$ are of orders which exceed two.

It is possible that different choices of non-invariant operators of order two might affect the order of H_m. We suppose that the operations were effected in such a manner that the order of H_m is as large as possible. Such an H_m can be obtained by a finite number of operations since g is finite. The index of H_m under G cannot be less than $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}$, as the order of H_m is a power of 2. Since at least half of the operators in $H_x - H_{x+1}$ are of orders which exceed two, it is clear that the minimum number of operators whose orders exceed two in a group of order g is one-half the number of operators in $G - H_m$ when the index of H_m under G has its minimum value, viz., $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\lambda^{\alpha_\lambda}$. That is, *$N$ is the minimum number of operators whose orders exceed two in any group of order g.*

It remains only to determine all the possible groups of order g which contain exactly N operators whose orders exceed 2. In what follows it will be assumed that G satisfies this condition. If an operator of order two in $H_{m-1} - H_m$ were commutative with some operator whose order exceeds two in H_{m-1} their product would be in $H_{m-1} - H_m$ and would be transformed into its inverse by t_m. As t_m would transform into its

inverse the given operator whose order exceeds two as well as this product, it would have to be commutative with the operator of order two from $H_{m-1} - H_m$. Since this is impossible, it follows that all the operators of order two in $H_{m-1} - H_m$ are commutative with exactly 2^{α_0} operators in H_{m-1}* and that these operators constitute a subgroup which is conjugate with H_m under H_{m-1}.

Since all the subgroups of order 2^{α_0} are conjugate under H_{m-1} and one of them contains an operator which is non-invariant under H_{m-1}, all of these Sylow subgroups must have this property. If an operator occurs in two of them it must occur in all of them, since it is commutative with operators of H_{m-1} whose orders exceed two. All the non-invariant operators of order two in H_{m-1} must therefore transform every operator whose order exceeds two in H_{m-1} into its inverse. As one of the latter operators is transformed into its inverse by at least half the operators in $H_{m-1} - H_m$ and by the operators of H_m which are not commutative with every operator of H_{m-1}, it must be transformed into its inverse by just half the operators of H_{m-1}. Hence H_m contains just 2^{α_0-1} operators which are invariant under H_{m-1}.

There are as many operators in H_{m-1} that are commutative with one of its operators whose order exceeds two as there are of those which transform this operator into its inverse. As the latter includes half the operators of H_{m-1} the former set must be composed of the remaining half. Hence every operator whose order exceeds two in H_{m-1} is commutative with each of the other operators in H_{m-1} which have this property. From this it follows that all the operators of odd order in H_{m-1} together with identity constitute an abelian subgroup, and that any non-invariant operator of order two together with this abelian subgroup generate an invariant subgroup in which just half its operators are of order two. This subgroup is invariant since it includes all the conjugates of its operators of order two. Hence H_{m-1} *is the direct product of a subgroup in which just half the operators are of order two and the subgroup of order* 2^{α_0-1} *formed by the invariant operators of* H_{m-1}.

We shall now prove that the order of H_x is divisible by $p_1^{\alpha_1}$ whenever it is divisible by p_1. Suppose that $p_1^{\alpha_i}$ is the highest power of p_1 which divides the order of H_x and that the order

* Every operator of order two must be found in one of the Sylow subgroups of order 2^{α_0}, and all of these subgroups are similar to H_m.

of H_{x-1} is divisible by $p_1^{\alpha'+1}$. Hence H_{x-1} contains a subgroup of order $p_1^{\alpha'+1}$ which has $p_1^{\alpha'}$ operators in H_x and the remaining operators in $H_{x-1} - H_x$. All the operators of H_x are commutative with t_x and all the operators whose orders exceed two in $H_{x-1} - H_x$ are transformed into their inverse by t_x. As this is impossible* it follows that the order of H_x is divisible by $p_1^{\alpha_1}$ whenever it is divisible by p_1.

Our next object is to prove that H_{m-1} is identical with G. This will be proved by showing that the opposite hypothesis leads to an absurdity. Suppose that the order of H_{m-1} is $2^{\alpha_0} p_1^{\alpha_1} \cdots p_{\lambda'}^{\alpha_{\lambda'}}$, $\lambda' < \lambda$. Hence the order of H_{m-2} is divisible by primes which do not divide the order of H_{m-1}. The operator t_{m-1} of order two which is invariant under H_{m-1} but not under H_{m-2} is found in the subgroup K of H_{m-1} which is composed of its 2^{α_0-1} invariant operators. All the operators whose orders exceed two in $H_{m-2} - H_{m-1}$ are transformed into their inverses by t_{m-1}. As the number of conjugates of t_{m-1} under H_{m-2} is the index of H_{m-1} under H_{m-2} it follows that the operators in $H_{m-2} - H_{m-1}$ cannot be transformed into more different operators to obtain their inverses than the given index. Hence these operators cannot include more operators of odd order than this index, since two distinct operators of odd order must be multiplied by two distinct operators in order to obtain products which are the inverses of the first two operators.

It is now easy to prove that the Sylow subgroup of order $p_1^{\alpha_1}$ contained in H_{m-1} is transformed into itself by operators of odd order in $H_{m-2} - H_{m-1}$. This is true when this is the only subgroup of order p^{α_1} in H_{m-2}. It is also true when this Sylow subgroup has less conjugates under H_{m-2} than the index of H_{m-1} under H_{m-2}, since this index involves no primes which divide the order of H_{m-1}. The remaining cases are when this Sylow subgroup has as many conjugates under H_{m-2} as this index and some of these Sylow subgroups have common operators or no two of them have common operators. The former of these two cases is excluded by the last footnote, since the Sylow subgroup in H_{m-1} would have operators in common with some other Sylow subgroup. The latter of the two cases is excluded by the fact

*The truth of this statement follows from the elementary theorem: If an operator transforms all the operators of a group which are not included in an invariant subgroup of index $k > 2$ into their inverses it transforms every operator of the group into its inverse and the group is abelian. Hence all the Sylow subgroups of odd order contained in G are abelian.

that the number of operators of odd order in $H_{m-2} - H_{m-1}$ cannot exceed the index of H_{m-1} under H_{m-2}.

Having proved that the assumption that the order of H_{m-1} is less than the order of G requires that the Sylow subgroup of order $p_1^{a_1}$ in H_{m-1} is transformed into itself by operators of odd order in $H_{m-2} - H_{m-1}$, it is now easy to see that this assumption leads to an absurdity. In fact, the theorem of the last footnote shows clearly that this subgroup of order $p_1^{a_1}$ cannot be transformed into itself by any operator of odd order in $H_{m-2} - H_{m-1}$, since it would be invariant in a group which would have only $p_1^{a_1}$ operators in common with H_{m-1} and the remaining operators of this group would be transformed into their inverses by t_{m-1}. Hence the theorem: *If a group of order g contains the smallest possible number of operators whose orders exceed two, the subgroup which is composed of all its operators which are commutative with one of the non-invariant operators of order two contains no operator whose order exceeds two.* Moreover, this subgroup is a Sylow subgroup, and just half of the remaining operators are of order two.

If we combine with this theorem the one stated above in regard to H_{m-1}, we have a complete determination of all the possible groups of order g in which the number of operators whose orders exceeds two is a minimum. To construct these groups it is only necessary to construct all the possible abelian groups of order $p_1^{a_1}p_2^{a_2}\cdots p_\lambda^{a_\lambda}$ and add to each of them an operator of order two which transforms each of its operators into its inverse. The groups thus obtained may be multiplied directly into the group of order 2^{a_0-1} which contains only operators of order two in addition to identity. Hence there are as many such groups as there are abelian groups of order $p_1^{a_1}p_2^{a_2}\cdots p_\lambda^{a_\lambda}$.

The preceding considerations show that the minimum number of operators whose orders exceed two in any group with the single exception of the abelian group of order 2^a and of type $(1, 1, 1, \cdots)$ is one fourth of the order of the group. If more than one fourth of the operators have orders which exceed two, at least one third of the operators have this property. When the order of the group is a power of 2, all the possible ratios between the number of the operators whose orders exceed two and the order of the group have been determined on the hypothesis that this ratio is less than one half. It is clear that N, in the present article, has for its limits one third and one half respectively, — the former being attained, while the latter is not attained.

NOTE ON THE ORIENTATION OF A SECANT.

BY PROFESSOR L. D. AMES.

(Read at the Preliminary Meeting of the Southwestern Section of the American Mathematical Society, December 1, 1906.)

In a previous paper * the orientation of curves and their tangents is briefly discussed. For certain problems in the theory of the potential function it is desirable to be able to define uniquely the positive sense along a secant through two variable points of an open smooth curve in such a way that the angle α between the positive direction of the secant and a fixed line shall be a continuous function of the two variable points. Such a definition will include the tangent as a special case. A natural method would be to define the positive sense uniquely and then prove that α is continuous. Another method would be to define the positive sense of the secant for one position and then define the positive sense in every other position so that the angle α shall be continuous. Either method involves somewhat troublesome considerations if the angle is defined by means of any one of the inverse trigonometric functions. The method of defining an angle used in the paper above mentioned proves useful in this case.

Given the open curve

$$x = x(t), \quad y = y(t), \qquad t_0 \leqq t \leqq T,$$

where x, y, dx/dt, dy/dt are single-valued and continuous in the interval (t_0, T), and where

$$x(t_1) \neq x(t_2), \quad y(t_1) \neq y(t_2) \quad \text{when } t_1 \neq t_2.$$

Define a secant through the points t_1, t_2 as follows:

$$X = \epsilon\lambda\rho + x(t_1), \quad Y = \epsilon\mu\rho + y(t_1),$$

where ϵ is arbitrarily chosen either $+1$ or -1, ρ is the variable parameter, and

* "An arithmetic treatment of some problems in analysis situs," *Amer. Jour. of Mathematics*, Oct., 1905, p. 349.

$$\lambda = \frac{x(t_2) - x(t_1)}{t_2 - t_1}, \qquad \mu = \frac{y(t_2) - y(t_1)}{t_2 - t_1} \quad \text{when} \quad t_1 \neq t_2$$

and

$$\lambda = \frac{dx_1}{dt}, \qquad \mu = \frac{dy_1}{dt}, \qquad \text{when} \quad t_1 = t_2.$$

Define the *positive sense* along this line to be the direction of increasing ρ. This is uniquely defined for any pair of points in the interval (t_0, T) since λ and μ are unchanged by interchanging t_1 and t_2.

Define an angle α as follows:

$$\sin\alpha = \epsilon k\mu, \quad \cos\alpha = \epsilon k\lambda, \quad k = (\lambda^2 + \mu^2)^{-\frac{1}{2}}.$$

Then α is an infinitely many-valued function of t_1 and t_2, its values for any given pair of values of t_1 and t_2 differing by multiples of 2π. If one of these values α' be assigned to a particular pair t_1', t_2', then from the possible values of α one and only one single valued continuous function can be chosen which takes the value α' at t_1', t_2'.*

UNIVERSITY OF MISSOURI,
November 5, 1906.

ON EULER'S ϕ-FUNCTION.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, December 28, 1906.)

THE object of the present note is the demonstration of certain very elementary propositions concerning Euler's ϕ-function of a number.

I. *The relation $\phi(m) = n$, a given number, is never uniquely satisfied for any given value of n.* That is, there is always more than one value of m for every possible value of n.

If any solution is $m =$ an odd number, then the given relation is satisfied by $2m$ also. Likewise, if m is twice an odd number, we may show that $m/2$ will also satisfy the relation.

*Cf., e. g., Stolz, Differential-Rechnung, vol. 2, p. 15–20.

Hence, if there is a unique solution, m is a multiple of 4; say $m = 4\mu$. Now n is even; say $n = 2\nu$. Then we have

$$\phi(4\mu) = 2\nu.$$

Hence

$$\phi(2\mu) = \nu.$$

Then in a manner similar to the above we may show that μ and ν are both even. By continuing the process step by step we are able to show that a unique solution cannot exist unless both m and n are powers of 2. It remains therefore to show that this cannot give a unique solution. Let $n = 2^\alpha$. Then

$$\phi(m) = 2^\alpha$$

is satisfied not only by $m = 2^{\alpha+1}$ but also by $m = 2^a(2^b + 1)(2^c + 1)\cdots$ in every way in which $a, b, c, \cdots$ can be so chosen that $a + b + c + \cdots = \alpha + 1$, $a \neq 0$; or $b + c \cdots = \alpha$, $a = 0$; and $2^b + 1, 2^c + 1, \cdots$ shall be different primes. If $\alpha \geqq 3$, one such solution is always $a = \alpha - 2$, $b = 1$, $c = 2$. An examination for the smaller values of α shows that no unique solution exists in these cases. Hence the proposition

II. *The equation* $\phi(m) = 2^n$ *has just* $n + 2$ *solutions when* $n + 2 \leqq 33$; *but just* 33 *solutions for* $n =$ *any number from* 32 *to* 255.

An odd solution evidently requires

$$m = (2^\alpha + 1)(2^\beta + 1)(2^\gamma + 1)\cdots,$$

where $\alpha + \beta + \gamma + \cdots = n$ and each factor is prime. Since $2^x + 1$ is prime* for $x = 1, 2, 4, 8, 16$, but not for $x = 32, 64, 128$ nor any value of x not of the form 2^p, it is clear that $\phi(m) = 2^n$ has one and but one odd solution for every value of n up to $n = 31$. Also twice every even value of m which satisfies $\phi(m) = 2^n$ will satisfy the equation when the second member becomes 2^{n+1}. Hence the number of solutions is increased by one when n is so increased up to $n = 31$; but beyond that up to $n = 255$ the number of solutions remains constant; for there is then no solution except those given by twice each solution for the preceding value of n. Up to $n = 31$ it is easy to see that the number of solutions is $n + 2$; then from this point onward to $n = 255$ the number remains constant and is 33.

*See BULLETIN, June 1906, p. 449.

It will be noticed that the first value of 2^n to which there corresponds no odd solution in m is 2^{32}. This is the smallest value of $\phi(m)$ known to the writer to have no odd solution in m.

III. COROLLARY. *The equation* $\phi(m) = 2^n$ *has (only) one odd solution when* $n \leqq 31$; *otherwise no odd solution at all up to* $n = 255$. *Also it has evidently no other odd solution except for such values of* n *as make* $2^n + 1$ *prime.*

IV. *All the solutions of the equation* $\phi(m) = 4n - 2$, $n \neq 1$, *are of the form* p^α *and* $2p^\alpha$, *where* p *is a prime of the form* $4s - 1$.

Now $m \neq 4$. Then if m contains the factor 4 it is evident that the equation is not satisfied. Neither is it satisfied if m contains two odd primes. Therefore the only values left are of the form p^α and $2p^\alpha$. Moreover p must be a prime of the form $4s - 1$; for otherwise the equation is not satisfied. (There may evidently be more than one p which furnishes such a solution. A case in point is $\phi(m) = 18$, which has the solutions $m = 19$, 27, 38, 54.)

V. *If* p *is of the form* $4s - 1$ *and* $\phi(m) = p^\alpha(p - 1)$ *has but the two solutions* $m = p^{\alpha+1}$, $2p^{\alpha+1}$, *then the relation* $\phi(m) = 2p^\alpha(p - 1)$ *has an odd solution.* (α belongs to the series 0, 1, 2, $\cdots$.)

For one solution of the latter is $m = 4p^{\alpha+1}$. There is no other solution in which m is a multiple of 4; for then there would correspond to that a third solution for $\phi(m) = p^\alpha(p - 1)$. But $\phi(m) = 2p^\alpha(p - 1)$, by proposition I, has more than one solution. Hence it is easy to see that it has both an odd solution and another twice that one.

PRESBYTERIAN COLLEGE,
ANNISTON, ALA.

SHORTER NOTICES.

The Theory of Determinants; in the Historical Order of Development. Second edition, revised. By THOMAS MUIR. London, Macmillan and Co., 1906. xii + 491 pp.

THE present work "is intended for the student who aims at acquiring such a knowledge as can only be got by a study of the subject in the historical order of its development, for the investigator who is specially interested in this branch of mathematics and wishes to become acquainted with the various lines of attack opened up by previous workers, and for the general working mathematician who requires guide books and books of reference concerning special domains."* It covers the period from the beginning of determinants in 1693 to the close of 1841. Jacobi's long and exhaustive memoirs published in volumes 27 and 29 of *Crelle's Journal* are also considered in connection with skew determinants, although they appeared three or four years after the close of the period under consideration.

About one-third of the space is devoted to quotations from the works of the authors under consideration. These quotations enable the reader to form an independent judgment in reference to questions of priority and are especially valuable to those who do not have access to large libraries. Each new theorem of importance is numbered in Roman figures, and if the same result is either generalized or obtained in a different way by a subsequent writer it is marked among the contributions of the latter with the same Roman figure followed by an Arabic numeral.

Two very useful tables are inserted, showing the advance of the subject from 1693 to 1812 and from 1813 to 1841 respectively. There is also an index to the numbered results, but there is no general subject index. Its value as a reference book would have been greatly enhanced by such an index, especially since the chronological order of arrangement made it frequently impossible to bring together the material relating to the same topic.

The book is divided into two parts. Part I, in ten chapters, treats of general determinants and occupies considerably more

* From the preface.

than half the space. Complete chapters are devoted to the memoirs appearing in each of the years 1812 and 1841, while only one is given to those which appeared in the eighty-seven years from 1693 to 1779. Part II is devoted to special determinants which began to be studied, though without the present notation, by Lagrange and Gauss as early as 1773 and 1801 respectively. One chapter is devoted to each of the following subjects: axisymmetric determinants, alternants, jacobians, skew determinants, orthogonants.

Chapter XVI treats the miscellaneous special forms which prior to 1841 had received but little attention. The most fertile originator of such forms was Wronski, a poor Polish enthusiast whose egotism and wearisome style tended to attract few followers. The chapter is mainly devoted to the four special forms of determinants studied by Wronski, three of which were introduced by him. The other writers considered in this chapter are Scherk, Schweins, Jacobi, and Sylvester, the first two in relation to extensions of Wronski's results, the last two in relation to special forms of determinants arising from Bezout's method of eliminating the unknown between two equations of the nth degree and from Sylvester's dialytic process.

The last chapter consists of a single paragraph, devoted to a retrospect on special forms. Two other brief chapters (V and X) also are devoted to retrospects. In the former of these the fact is emphasized that nearly all of the earlier theorems are due to French writers, while the latter notes the rapid increase of interest after 1812 — especially in Germany, where Jacobi and Schweins took the most prominent part in the development of the subject. This spread of interest was largely due to the study of determinants whose elements have special forms. The beauty of some of these results made the notation more attractive and it spread rapidly into common use. Its great advantages as a thought saver, replacing the repeated thought tensions by a single demonstration, soon won for it universal favor, and, in some instances, undue prominence.

As the author of the present volume has published the most complete bibliography of determinants* and has contributed towards its advancement during a period of more than twenty years, we naturally expect the work to have the charm of ripe

* *Quar. Jour. of Mathematics*, vols. 18, 21, 36. The bibliography includes 1,744 titles and covers the periods from the earliest writings on the subject down to the close of the nineteenth century.

scholarship. This expectation is enhanced by the fact that the author confines himself to an early period and that the present volume is a second edition of a valuable work that has been before the public for sixteen years. The typographical errors are few and insignificant. The most serious noticed occurs on page 4, where it is stated that the earlier edition appeared in 1900, instead of 1890.

In 1838 Liouville demonstrated a property of the special forms which were afterwards called wronskians by Muir (1881) and later writers. According to Anissimov * this seems to be the first important result relating to these useful forms, but the paper is not mentioned by our author although it falls within the period covered. Neither is it included in the bibliography cited above.

The present volume is almost twice as large as the first edition and includes a number of papers not there mentioned. The greatest difference between the two editions is in the facts that the papers relating to special forms are brought together in the new edition and that a table showing the advance of determinants from 1813 to 1841 has been added. The use of smaller type for quotations has greatly improved the appearance of the text. The completeness and attractiveness of the book combine to make it indispensable to the student of determinants and their history. G. A. MILLER.

A First Course in Analytical Geometry, Plane and Solid, with Numerous Examples. By CHARLES N. SCHMALL. New York, Van Nostrand, 1905. 8vo. 7 + 318 pp.

ON the whole, this little book resembles so closely the ordinary text on the subject, both as to material and treatment, that a detailed account is scarcely necessary. Mention of the few points in which it does depart from the general type of text will be sufficient. In the chapter on loci, the usual examples from physics, economics, statistics, etc., are omitted, and attention is confined exclusively to purely geometric problems. This omission will probably be regarded as a fault or a merit according to one's views on the recently discussed relations of mathematics to physics and other subjects. The chapter on the circle precedes that on the transformation of coordinates, reversing

* Encyclopédie des Sciences Mathématiques, Tribune publique I, 1906, p. 3.

the usual order. A short chapter on confocal conics is inserted, while the treatment of higher plane curves and of solid geometry are unusually brief.

The most interesting novel feature is the early introduction of the determinant notation and its continued use throughout the book. In the first equation of the straight line (*i. e.*, in terms of the coordinates of two known points) the determinant form is given side by side with the explicit equation. Even if the student has had no previous treatment of the determinant, a few minutes of class-room explanation will enable him to grasp this simple form and the continued use will certainly give him that realizing sense of the geometric value of this notation which he too often fails to get in his formal course in algebra. E. B. COWLEY.

Lehrbuch der analytischen Geometrie. Erster Band: *Geometrie in den Grundgebilden erster Stufe und in der Ebene.* Von L. HEFFTER und C. KOEHLER. Leipzig, Teubner, 1905. 8vo. 16 + 526 pp.

THIS volume is in strong contrast to the book just noticed. While the authors aim to make the treatment elementary, they wish to give the student an introduction, at least, to the modern methods of relating the various parts and kinds of geometry into one comprehensive whole. They endeavor to follow the way suggested by Cayley's Sixth memoir on quantics and sketched by Klein in his Erlangen Programm. They consider that this can be done only by the early introduction of the transformation group, proceeding from the projective group to its subgroup, the affine, and then to its subgroup, the so-called "äquiform"; and not in the inverse order. This first volume contains the geometry in all spaces of one dimension, and in the plane. The second will be devoted to the finite "Bündel" and to ordinary space. In each case, the procedure is from projective to affine and then to æquiform geometry.

A condensed account of the contents will be useful in giving some idea of the ground covered. After an introductory chapter devoted to definitions and a few general considerations, there follows the first part, which is on geometry in spaces of one dimension. The first chapter relates to projective and affine geometry in the point range. The next takes up the quadratic equation and the point pair and its involution. This part is concluded by a consideration of the projective and

æquiform geometries of the real pencils. The second section begins, under the heading of geometry in a plane, with a chapter on projective geometry in the plane and coördinates. The second chapter considers the principle of duality and projective theorems concerning points and straight lines. In the third, the projective and the reciprocal transformations of the plane are discussed. Then follow two chapters on the elements of the affine and æquiform geometries in the plane, and one on the general projective properties of curves of the second order and second class. After the projective classification of conics and the consideration of polarity in reference to conics, come three chapters treating of pencils and ranges of conics, the first in projective and the other two in affine geometry. The twelfth chapter takes up the æquiform geometry of conics. The next two deal with principal axes and foci and focal properties. In the last chapter is found the æquiform geometry of systems of conics (ranges and pencils). The book is concluded by an appendix on determinants. No knowledge of the calculus is presupposed.

The original plan was to begin with a systematic treatment of the axioms and base all subsequent work upon them. However, consideration for the mathematical immaturity of the students has led to some modifications; for example, the lengthy and abstract discussion of the axioms is omitted and a few theorems are inserted without proof. One naturally asks about the treatment of cross ratio. It is defined (page 36) in terms of the ratios of segments which have been measured in the ordinary way. With this metrical definition, it is used in projective geometry. In the footnotes, however, references are given to books in which the student will find a purely projective treatment.

The question that arises as one reads is that of the advisability of beginning with the spaces of one dimension. Will the beginner really grasp the ideas, are they concrete enough for his full comprehension, or will he fail to appreciate the purpose of the discussion, even though he may follow the separate details? In other words, it is a logical order; but, is it psychological? These are questions that can, perhaps, be answered only after one has attempted to use the book as a text. We are too ready to assume that a method of approach that is unusual will necessarily be difficult for the student. The style throughout is clear and simple, as well as stimulat-

ing and suggestive. The general plan seems admirable; and the student should have mastered in the end not only the usual collection of time-honored facts about conics, but a few of the well-known theorems such as Desargues's, Brianchon's, and Pascal's, as well as an introductory idea, at least, of that most important geometric concept, — the group of all projective transformations.

E. B. COWLEY.

Tratado de las Curvas Especiales Notables. By F. GOMES TEIXEIRA. Madrid, "Gaceta de Madrid," 1905, ix + 632 pp.

THIS volume had its inception in a prize problem proposed in 1892, and repeated in 1895, by the Royal Academy of Sciences of Madrid, requiring "An orderly list of all the curves of every kind to which definite names have been assigned, accompanying each with a succinct exposition of its form, equations and general properties, and with a statement of the books in which, or the authors by whom, it was first made known." This programme our author has closely adhered to except in one particular. To attempt to give the properties of all such curves would be extremely difficult and would make the resulting work unwieldy, he has therefore wisely limited himself to a list of over one hundred curves so selected as to include almost all of especial importance.

This treatise and Loria's work, "Spezielle algebraische und transcendente ebene Kurven," which appeared a little earlier, cover almost the same field. Both authors seem to have taken their suggestion from the theme of the Royal Academy. Teixeira, however, has followed that programme more closely. Loria's work is arranged in a more satisfactory manner and is somewhat more advanced in treatment. Teixeira's has the advantage of giving a considerable discussion of space curves.

The first two chapters (98 pages) of Teixeira's treatise are devoted to a detailed exposition of the properties of the most important cubic curves. In the third, fourth, and fifth chapters (158 pages) he treats of quartics and in the sixth chapter (68 pages) of algebraic curves of order higher than the fourth. He considers in the seventh chapter (39 pages) a number of transcendental curves, most of them of considerable physical importance. The spirals are considered in Chapter VIII (47 pages), the parabolas and hyperbolas $y = a^{1-k}x^k$ in Chapter IX (10 pages), and the cycloidal curves in Chapter X (56

pages). Chapter XI (45 pages) is devoted to the discussion of several plane curves of lesser importance. An exposition of the properties of a number of space curves is given in Chapters XII and XIII (72 pages). These chapters are not, however, up to the standard of the rest of the book. Chapter XIV (13 pages) is devoted to polhodes and herpolhodes.

The treatment is, throughout, quite elementary, and can be followed by anyone with a knowledge of analytics and calculus. The book is not, however, suitable for a student seeking a systematic treatment of the theory of the higher curves, as its aim is the consideration of noteworthy curves and not the systematic exposition of curves in general. It is a treatise, not on curve theory, but on particular curves.

The method of exposition, whenever practicable, is as follows: the rectangular and polar equations of the curve are given, the form of the curve is derived from the equations and a figure of the curve is shown. The most interesting geometric properties of the curves are then deduced, the parametric equations — when the curve is unicursal — derived and the integrals for the length of arc and the area of the curve obtained.

The book is metrical both in viewpoint and in method. Trilinear coordinates are, however, used a few times and line coordinates several times but not in such a manner as to necessitate a previous knowledge of those subjects. It is to be regretted that the author does not enter into the projective theory at least enough to show the projective interrelations, and occasionally the projective identity, of some of the curves considered.

The content and method of the book are such as to make it especially valuable for engineers and others not specialists in geometry, but geometers also will find in it a valuable collection of information on particular curves. The Spanish text presents little difficulty to one who can read any Romance language with facility. C. H. SISAM.

The Scientific Papers of J. WILLARD GIBBS. Volume 1, *Thermodynamics*, xxviii + 434 pp., with portrait; volume 2, *Dynamics, vector analysis and multiple algebra, electromagnetic theory of light, etc.*, ix + 284 pp. London, Longmans, Green, and Co., 1906. Large 8vo.

SHORTLY before the death of Professor J. Willard Gibbs he had decided to yield to requests from various sources and to

reprint his principal papers on thermodynamics with some additional chapters, upon which he was then at work. After his sudden death had cut short his plans, his family resolved to fulfil his intention, as far as might be, by printing, under the editorial direction of Professor Bumstead and Dr. Van Name, a complete edition of his published papers * with such additions as could be made out from his brief and meager notes.

The first volume of the Scientific Papers contains the collection of articles on thermodynamics with seventeen supplementary pages of unpublished fragments which represent a first draft of a part of the material that Professor Gibbs was preparing. Among the subjects which he was intending to treat were:

I. On the values of potentials in liquids for small components.
II. On the fundamental equations of molecules with latent differences.
III. On the fundamental equations for vanishing components.
IV. On the equations of electric motion.
V. On the liquid state, $p = 0$.
VI. On entropy as mixed-up-ness.
VII. Geometric illustrations.
VIII. On similarity in thermodynamics.
IX. Cryohydrates.

Of these, his notes contained but the partial treatment of I and IV, although VI may be regarded as somewhat discussed in chapters XII to XV of the Statistical Mechanics. One term which attracts the attention of the reader as probably something of great importance is *latent differences*. Unfortunately the notes left by Professor Gibbs do not permit of any definite conclusions as to what he meant. This must be a cause of deep regret to the scientific world — a feeling which may be somewhat ameliorated by the fact that at last the complete works of Willard Gibbs are easily available for reading and reference.

The second volume contains all the other papers, among which may especially be noted the original pamphlet on Vector Analysis, privately printed in 1881–4; an unpublished letter

* With the exception of his two books in the Yale Bicentennial Series: Vector Analysis (1901), edited by E. B. Wilson, and Statistical Mechanics (1902), Charles Scribner's Sons, New York City.

on the use of vector methods in the determination of orbits, written to Hugo Buchholz; and a short abstract, bearing the date 1884, of a communication to the American association for the advancement of science on the subject of statistical mechanics, which shows that even at that early date the author was tolerably far advanced on the investigations which he published only in 1902. It is regrettable that some account of the later researches on multiple algebra were not available for publication, in particular the matter concerning double products which constitutes an important advance in the general theory. Professor Gibbs seems to have kept this, like many other ideas certainly in an advanced state of development, in his mind rather than on paper. For this reason his published work which has now appeared contains practically all that the world can ever hope to know of his investigations.

E. B. WILSON.

Leibnizens Nachgelassene Schriften physikalischen, mechanischen und technischen Inhalts. Herausgegeben von DR. ERNST GERLAND. Leipzig, Teubner, 1906, vi + 256 pp.

THE volume contains hitherto unpublished papers of Leibniz on physics and mechanics, mostly written during the interval 1670 to 1686, or between the 24th and 40th year of his life. They are partly in Latin, partly in German and partly in French. The editor has supplied historical and explanatory notes. These articles are of interest, not only as showing that Leibniz sometimes touched upon questions which have since become of fundamental importance, such as the conservation of energy, but also as showing the progress of his ideas and his method of work.

FLORIAN CAJORI.

NOTES.

THE *American Mathematical Monthly* will hereafter be edited by Professors B. F. FINKEL and H. E. SLAUGHT, with the cooperation of Professor L. E. DICKSON. Since August, 1906, the *Monthly* has been published under the auspices of the University of Chicago.

AT the meeting of the London mathematical society held on December 13, the following papers were read: By C. S. JACKSON, "On the form of a surface of a searchlight reflector"; by L. F. RICHARDSON, "The potential equation and others with function given on the boundary"; by J. MERCER, "On the limits of real variants"; by E. W. BARNES, "The asymptotic expansion of integral functions defined by generalized hypergeometric series"; by P. A. MCMAHON, "The diophantine equation $x^n - Ny^n = z$.

AT the meeting of delegates of associations of teachers of mathematics and the natural sciences held at Columbia University on December 27, it was decided to organize an American federation of such associations. A committee, with Professor T. S. FISKE as chairman and Professor C. R. MANN as secretary, was appointed to take charge of the preliminary steps.

A FIFTH installment of the second part of volume 10 of the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, containing pages 1073–1392 of Burkhardt's Bericht über die Theorie der oscillierenden Funktionen has just appeared from the press of Teubner, in Leipzig. This does not complete the volume, as promised, however, but treats of English advances in mathematical physics up to the time of Kelvin, and the subsequent development of the theory of heat. It is proposed in the next installment to treat of each recent branch separately.

THE firm of B. G. Teubner in Leipzig announces that the following volumes of the *Mathematische Annalen*, the first edition of which was exhausted and long out of print, have been reprinted and can now be obtained for 28 marks each: volumes 3, 4, 5, 6, 7, 32, 33, 50, 51, 52, 53.

A BUST of the late Professor GUIDO HAUCK has been unveiled in the court of the main building of the technical school at Charlottenburg.

A TABLET has been placed in Memorial Hall of Yale University in memory of ELIAS LOOMIS, LL.D., professor of natural philosophy and astronomy from 1860 to 1889.

AT the Collège de France Dr. P. BOUTROUX has been appointed to the special lectureship established a few years ago by M. Peccot under the condition that the appointee should be under thirty years of age. The tenure of office is three years. The preceding holders of the office are Professors Baire, Borel, and Lebesgue and Dr. Servant.

PROFESSOR A. E. H. LOVE, of Oxford University, has been elected president of Section A (mathematics and physics) for the Leicester meeting of the British Association for the advancement of science.

PROFESSOR L. BIANCHI, of the University of Pisa, has been elected a member of the Royal institute of Venice.

PROFESSOR W. FIEDLER, of the technical school at Zürich, has been elected corresponding member of the academy of sciences at Munich.

MR. T. FRIESENDORFF, docent in the University at St. Petersburg, has been appointed associate professor of applied mathematics at the technical school at St. Petersburg.

PROFESSOR M. LERCH, of the University of Freiburg, Switzerland, has been appointed professor of mathematics at the Bohemian technical school at Brünn.

PROFESSOR V. OPPOLZER, of the University of Innsbruck, has been promoted to a full professorship of astronomy at the same institution.

PROFESSOR F. STREINTZ has been promoted to a full professorship of mathematics at the technical school at Graz.

PROFESSOR H. DE VRIES, of the technical school at Delft, has been appointed professor of mathematics at the University of Amsterdam.

DR. N. NIELSEN, of the University of Kopenhagen, has been promoted to an associate professorship of mathematics, and elected foreign member of the academy of sciences of Halle.

DR. H. DE S. PITTARD has been appointed assistant lecturer in mathematics at King's College, London.

MR. G. G. CHAMBERS has been appointed instructor in mathematics at the University of Pennsylvania.

DR. A. W. PANTON, lecturer in mathematics at Trinity College, Dublin, and joint author of Burnside and Panton's Theory of Equations, died December 18, at the age of 60 years.

DR. THOMAS HARRISON, professor of mathematics and later chancellor of the University of New Brunswick, died September 18, 1906, at the age of 68 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BENDT (F.). Differential- und Integralrechnung. 3te Auflage. Leipzig, 1906. 8vo. Cloth. M. 3.00

BÜRKLEN (O. T.). Aufgabensammlung zur analytischen Geometrie des Raumes. Leipzig, Göschen, 1906. 16mo. 98 pp. Cloth. (Sammlung Göschen.) M. 0.80

BURKHARDT (H.). Funktionentheoretische Vorlesungen. 2te Auflage. Vol. II: Elliptische Funktionen. Leipzig, Veit, 1906. 8vo. 16 + 373 pp. M. 10.00

CAMPBELL (D. F.). A short course on differential equations. New York, Macmillan, 1906. 12mo. 8 + 96 pp. Cloth. $0.90

EGGENBERGER (J.). Beiträge zur Darstellung des Bernoullischen Theorems der Gammafunktion und des Laplaceschen Integrals. 2te Auflage. Jena, Fischer, 1906. 8vo. 79 pp. M. 2.50

FIELDS (J. C.). Theory of the algebraic functions of a complex variable. Berlin, Mayer & Müller, 1906. 8vo. 7 + 186 pp. M. 12.00

FUCHS (L.). Gesammelte mathematische Werke. Herausgegeben von R. Fuchs und L. Schlesinger. Vol. II: Abhandlungen 1875–1887, redigiert von L. Schlesinger. Berlin, Mayer & Müller, 1906. 8vo. 10 + 487 pp. M. 30.00

GRÜNBAUM (H.). Lehr- und Uebungsbuch der Differenzialrechnung für mittlere technische Lehranstalten, Realgymnasien, Oberrealschulen u. s. w., sowie zum Selbststudium. 2te, verbesserte Auflage. Würzburg, Frank, 1907. 8vo. 8 + 119 pp. M. 1.75

KOZÁK (J.). Grundprobleme der Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Vol. I. Wien, 1906. 8vo. M. 11.00

LAURENT (H.). La géométrie analytique générale. Paris, Hermann, 1906. 8vo. 7 + 153 pp.

NIELSEN (N.). Theorie des Integrallogarithmus und verwandter Transzendenten. Leipzig, Teubner, 1906. 8vo. 6 + 106 pp. M. 3.60

PINCHERLE (S.). Algebra complementare. Parte I: Analisi algebrica. 2a edizione. Milano, Hoepli, 1907. 16mo. 8 + 174 pp. (Manuali Hoepli.) L. 2.00

RE (A. DEL). Lezioni sulle forme fondamentali dello spazio rigato, sulla dottrina degli immaginari e sui metodi di rappresentazione nella geometria descrittiva. Napoli, Alvano, 1906. 8vo. 4 + 85 pp. L. 3.50

REISKY. Zur Einführung in die geometrische Analysis. Leobschütz, 1906. 4to. 11 pp. M. 1.50

REYE (T.). Die Geometrie der Lage. Vorträge. Abteilung II. 4te, umgearbeitete und vermehrte Auflage. Stuttgart, Kröner, 1907. 8vo. 8 + 335 pp. M. 10.00

SCHLESINGER (L.). See FUCHS (L.).

SCHRÖDER (R.). Aufgaben der Quadratur der Kegelschnitte. Gross-Lichterfelde, 1906. 8vo. 19 pp. M. 1 50

VEGAS (M.). Tratado de geometría analítica. Vol. I. 2a edición. Madrid, Fortanet, 1906. 8vo. 6 + 688 pp. P. 15.00

II. ELEMENTARY MATHEMATICS.

AUGUST (E. F.). Vollständige logarithmische und trigonometrische Tafeln. 28ste Auflage. Leipzig, 1906. 8vo. 8 + 204 pp. Cloth. M. 1.60

BOREL (E.). Arithmétique (premier cycle) Paris, Colin, 1907. 16mo. 7 + 221 pp. (Cours de mathématiques rédigé conformément aux nouveaux programmes.) Fr. 2.50

BORK (H.). Mathematische Hauptsätze. Ausgabe für Realgymnasien und Oberrealschulen. Nach dem Tode des Verfassers herausgegeben von M. Nath. Teil I: Pensum der Unterstufe (bis zur Untersekunda einschliesslich). 5te, durchgesehene und den preussischen Lehrplänen von 1901 angepasste Auflage Leipzig, Dürr, 1907. 8vo. 252 pp. Boards. M. 2.50.

BUDINICH (A.). See HEIS (E.).

BÜTTNER (A.). Die Elemente der Buchstabenrechnung und Algebra. Nebst einem Anhange, enthaltend Logarithmentafeln für die Zahlen von 1 bis 10,000. Für den Schul- und Selbstunterricht. 18te Auflage. Neubearbeitet von W. Petri. Bielefeld, Velhagen & Klasing, 1907. 8vo. 8 + 287 pp. Boards. M. 4.00

DAVIDSON (J). Arithmetic and algebra. London, Hodder, 1906. 12mo. (Self-educator series.) 1s.

DURELL (C. V.). Elementary problem papers. London, Arnold, 1906. 12mo. 128 pp. Cloth. 1s. 6d.

FAILOR (I. N.). Plane and solid geometry. New York, Century Company, 1906. 12mo. 12 + 418 pp. Cloth. $1.25

FEIST (A.). Mathematische Aufgaben. Verzeichnis der bei den Reifeprüfungen der neunstufigen höheren Lehranstalten des Herzogtums Braunschweig 1892-1903 bearbeiteten Aufgaben. Abteilung II. Braunschweig, 1906. 4to. 15 pp. M. 1.00

FLECHSIG. See LUCZAK.

FRENCH (C. H.) and OSBORN (G.). Graphs, or the graphical representation of algebraic functions. 2d edition, revised and enlarged. London, Clive, 1906. 12mo. 136 pp. Cloth. 1s. 6d.

FUCINI (C.). Geometria piana e nozioni di geometria solida per le scuole secondarie inferiori. 8a edizione. Genova, Gioventù, 1907. 8vo. 143 pp. L. 1.50

FUSS (K.). Die wichtigsten Sätze der Planimetrie und Stereometrie. Für den Schul- und Selbstunterricht. 2te, stark vermehrte und vollständig umgearbeitete Auflage. Nürnberg, Korn, 1907. 8vo. 5 + 142 pp. M. 2.00

HARTL (H.). Lehrbuch der ebenen Trigonometrie. Für den Unterrichtsgebrauch und für das Selbststudium. Mit über 500 Uebungsbeispielen nebst deren Resultaten. 2te, umgearbeitete Auflage. Wien, Hölder, 1906. 8vo. 3 + 106 pp. Boards. M. 1.44

HECHT (C.). Lehrbuch der elementaren Mathematik. Als Ergänzung zu "Rechenbuch für Mädchenschulen" bearbeitet. Teil II: Arithmetik. Bielefeld, Velhagen & Klasing, 1906. 8vo. 6 + 156 pp. Boards. M. 1.80

HEIS (E.). Raccolta di esempi e quesiti di aritmetica ed algebra ordinati ad uso delle scuole secondarie. Versione di A. Budinich. 3a edizione. Torino, Loescher, 1907. 8vo. 4 + 387 pp. L. 4.00

JAAKSON (J.). Graphische Multiplikations-, Divisions-, Quadraten- und Wurzeltabelle. Blatt I: Für 1-, 2- und 3-stellige Zahlen. Berlin, 1906. Sq. 16mo. 2 pp. M. 0.50

LANNER (A.). Neuere Darstellungen der Grundprobleme der reinen Mathematik im Bereiche der Mittelschule. Berlin, Salle, 1907. 8vo. 8 + 192 pp. M. 3.00

LONDON matriculation mathematics papers. From January, 1896, to September, 1906. London, Clive, 1906. 12mo. 130 pp. Cloth. 1s. 6d.

LUCHERINI (R.). Elementi di aritmetica e geometria. Parte I, per gli alunni della IV classe elementare. Firenze, Pacetti, 1906. 16mo. 144 pp. L. 0.60

——. Parte II, per gli alunni della V classe elementare. Firenze, Pacetti, 1906. 16mo. 182 pp. L. 0.80

LUCZAK und FLECHSIG. Die Dreiteilung des Winkels. Neue, mathematisch genaue Lösung mit Hilfe des Luczakzirkels. Lissa, 1906. 8vo. M. 0.30

MARKS (C. I.). Mathematical questions and solutions from the *Educational Times*. New series, Vol. 10. London, Hodgson, 1906. 8vo. Cloth. 6s. 6d.

MERAY (C.). Nouveaux éléments de géométrie. 3e édition (réduction et refonte partielle de l'édition de 1903). A l'usage des écoles primaires supérieures, des écoles normales d'instituteurs, des écoles pratiques de commerce et d'industrie, des écoles professionnelles et de toutes les classes élémentaires dans les établissements d'enseignement secondaire. Dijon, Jobard, 1906. 8vo. 20 + 309 pp.

NATH (M.). See BORK (H.).

OSBORN (G.). See FRENCH (C. H.).

PALMIERI (F. S.). Elementi di aritmetica e di algebra per le scuole secondarie superiori. Parte I. Messina, Secolo, 1906. 16mo. 6 + 240 pp. L. 3.00

PASQUALI (P.). Geometria intuitiva, ad uso delle scuole elementari superiori, tecniche, normali e industriali. Lezioni di ritaglio geometrico. Parte III. Milano, Vallardi, 1906. 16mo. 39 pp. L. 0.50

PETRI (W.). See BÜTTNER (A.).

RIECHEMEYER. Das erste Jahr des planimetrischen Unterrichtes am Gymnasium. Gütersloh, 1906. 8vo. 20 pp. M. 1.50

ROEDER (H.). Trigonometrische und stereometrische Lehraufgabe der Unter-Sekunda (Prima der Realschulen). Sonderabdruck aus der Kamblyschen Planimetrie. 4te, vermehrte Auflage. Breslau, Hirt, 1906. 8vo. 4 + 49 pp. M. 0.60

RUIZ TAPIADOR (A.). Nociones y ejercicios de aritmética y geometría. Toledo, Serrano, 1906. 8vo. 108 pp.

SALOMON (A.). Leçons de géométrie, à l'usage de l'enseignement secondaire des jeunes filles. 3e édition. Paris, Vuibert et Nony, 1906. 16mo. 230 pp.

SCHÄCHER (W.). Das Wichtigste aus der geometrischen Formenlehre. 2te, verbesserte Auflage. Sternberg, Hitschfeld, 1906. 8vo. 39 pp. M. 0.36

SCHMID (A.). See SICKENBERGER (A.).

SIBIRANI (F.). Elementi di algebra per le scuole tecniche e normali, riveduti dal prof. Cesare Arzelà. Firenze, Le Monnier, 1906. 16mo. 151 pp. L. 1.50

SICKENBERGER (A.). Uebungsbuch zur Algebra. Abteilung I: Erste und zweite Stufe der Rechnungsarten einschliesslich der lineären Gleichungen mit einer und mehreren Unbekannten. 5te Auflage, bearbeitet von A. Schmid. München, Ackermann, 1906. 8vo. 5 + 106 pp. Boards. M. 1.35

TRUCCO (E.). Elementi di aritmetica ragionata, ad uso del ginnasio superiore. Treviglio, Messaggi, 1906. 8vo. 122 pp. L. 1.60

TUÑON DE LARA (M.). Lecciones de geometría. 1a edición. Toledo, Serrano, 1906. 8vo. 282 pp. P. 10.00

WINTER (W.). Stereometrie. Lehrbuch und Aufgabensammlung für Schulen. 4te Auflage. München, Ackermann, 1906. 8vo. 4 + 116 pp. M. 1.60

III. APPLIED MATHEMATICS.

AHRENS (C.). See LAUENSTEIN (R.).

ARNOLD (E.). Die Gleichstrommaschine. Ihre Theorie, Untersuchung, Konstruktion, Berechnung und Arbeitsweise. (In 2 Bänden.) Vol. I: Theorie und Untersuchung. 2te, vollständig umgearbeitete Auflage. Berlin, Springer, 1906. 8vo. 16 + 816 pp. Cloth. M. 20.00

EBERHARD (VON). See SABUDSKI (N.).

GIBBS (J. W.). The scientific papers of J. Willard Gibbs. (In two volumes.) Vol. I: Thermodynamics. 28 + 434 pp. Vol. II: Vector analysis and multiple algebra, electromagnetic theory of light, etc. 6 + 284 pp. New York, Longmans, 1906. 4to. Cloth. $9.00

HANDBUCH der Küstenvermessung, herausgegeben vom Reichs-Marine-Amt. 2 volumes. Berlin, Mittler, 1906. 8vo. 8 + 332 and 6 + 178 pp., 5 plates. Cloth. M. 5.00

HARBAUER (K.). Die praktische Geometrie (Feldmesskunst). 2tes Tausend. Wien, Stern, 1906. 8vo. 4 + 136 pp., 5 plates. M. 4.50

HENDERSON (R. S.). Railroad curve tables; containing a comprehensive table of functions of a one-degree curve, with correction quantities giving exact values for any degree of curve, together with various other tables and formulas, including radii, natural sines, cosines, tangents, cotangents, etc.; to which is added a method of finding any function of a curve of any degree or radius without a field book. New York, Engineering News Publishing Co., 1906. 16mo. 3 + 69 pp. Cloth. $1.00

HERRMANN (L.). See WEBER (A.).

HOSKINS (L. M.). A text book on hydraulics, including an outline of the theory of turbines. New York, Holt, 1906. 8vo. 5 + 271 pp. Cloth. $2.50

KESSLER (J.). Grundzüge der Mechanik. Kurzgefasstes Lehrbuch in elementarer Darstellung. Teil II: Dynamik fester Körper. Leipzig, Gebhardt, 1906. 8vo. 6 + 134 pp. M. 3.50

LAUENSTEIN (R.). Die Mechanik. Elementares Lehrbuch für den Schul- und Selbstunterricht, sowie zum Gebrauch in der Praxis. 7te Auflage, bearbeitet von C. Ahrens. Stuttgart, Kröner, 1907. 8vo. 7 + 217 pp. M. 4.40

LORENZ (H.). Neue Theorie und Berechnung der Kreiselräder. Wasser- und Dampfturbinen, Schleuderpumpen und Gebläse, Turbokompressoren, Schraubengebläse und Schiffspropeller. München, Oldenbourg, 1906. 8vo. 15 + 144 pp. Cloth. M. 8.00

LOVE (A. E. H.). Theoretical mechanics. An introductory treatise on the principles of dynamics. 2d edition. Cambridge, University Press, 1906. 8vo. 384 pp. Cloth. 12s.

MERRIMAN (M.). The strength of materials. A text-book for manual training schools. 5th edition, enlarged. New York, Wiley, 1906. 12mo. 156 pp. Cloth. $1.00

MOOSER (J.). Theoretische Kosmogonie des Sonnensystems. St. Gallen, Fehr, 1906. 8vo. 83 pp. M. 4.00

NERNST (W.). Theoretische Chemie vom Standpunkte der Avogadroschen Regel und der Thermodynamik. 5te Auflage. 1te Hälfte. Stuttgart, Enke, 1906. 8vo. Pp. 1–430. M. 10.00

NEUMANN (F.). Kugelkreise auf Mercators Seekarte, in elementarer Darstellung. Halberstadt, 1906. 4to. 20 pp. M. 1.50

SABUDSKI (N.). Die Wahrscheinlichkeitsrechnung, ihre Anwendung auf das Schiessen und auf die Theorie des Einschiessens. Uebersetzt von Eberhard. Stuttgart, 1906. 8vo. 8 + 438 pp., 2 plates, 7 tables. M. 8.80

SAINT-BLANCAT (D.). Action d'une masse intramercurielle sur la longitude de la lune. (Thèse.) Paris, Gauthier-Villars, 1906. 4to. 109 pp.

SCHLESSER (E.). Géométrie descriptive et géométrie cotée. (Classes de première et de mathématiques; préparation à l'Ecole navale, à l'Ecole spéciale militaire de Saint-Cyr, à l'Institut agronomique, etc.; programmes du 27 juillet 1905.) Paris, Delagrave, 1906. 8vo. 7 + 256 pp.

STEPHAN (P.). Die technische Mechanik. Elementares Lehrbuch für mittlere maschinentechnische Fachschulen und Hilfsbuch für Studierende höherer technischer Lehranstalten. Teil II: Festigkeitslehre und Mechanik der flüssigen und gasförmigen Körper. Leipzig, Teubner, 1906. 8vo. 8 + 332 pp. Cloth. M. 7.00

WEBER (A.). Theorie der Wechselströme. Vorträge. Bearbeitet und herausgegeben von L. Herrmann. 2te Auflage. Strelitz, Hittenkofer [1906]. 8vo. 9 + 64 pp. Cloth. M. 5.50

WINKELMANN (A.). Handbuch der Physik, herausgegeben von A. Winkelmann. Vol. I, 1te Hälfte: Allgemeine Physik. Leipzig, Barth, 1906. 8vo. 4 + 544 pp. M. 17.00

——. Vol. III, 2te Hälfte: Wärme. Leipzig, Barth, 1906. 8vo. Pp. 537–1178. M. 20.00

ZEITZ (R.). Handbuch der Nautik. Leipzig, 1906. 8vo. 11 + 306 pp., 11 plates. Cloth. M. 4.00

ZEUNER (G.). Technische Thermodynamik. 3te Auflage. Zugleich 5te, vollständig neu bearbeitete Auflage der "Grundzüge der mechanischen Wärmetheorie." Vol. II: Die Lehre von den Dämpfen. Leipzig, Felix, 1906. 8vo. 8 + 462 + 29 pp. M. 14.40.